Dr. Umayal Ramanathan College for Women,Karaikudi

(Accredited with B+ Grade by NAAC)

Department of Mathematics

LINEAR ALGEBRA

7BMA4C2

by

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SYLLABUS

COURSE CODE: 7BMA4C2

CORE COURSE - VIII – LINEAR ALGEBRA

Unit – I

Vector Spaces – Definition and examples – Subspaces – Linear Transformation – Span of a set.

Unit – II

Linear Independence – Basis and Dimension – Rank and Nullity.

Unit – III

Matrix of a Linear Transformation – Inner Product Space – Definition and examples – Orthogonality – Orthogonal complement.

Unit – IV

Algebra of Matrices – Types of Matrices – The inverse of a matrix – Elementary Transformations – Rank of a Matrix– Simultaneous linear equations.

Unit – V

Characteristic Equation and Cayley – Hamilton theorem Eigen values and Eigen Vectors, Bilinear forms – Quadratic forms.

Text Book:

1. Dr. S.Arumugam and Mr. A.ThangapandiIssac, Modern Algebra, SciTech Publications (India) Pvt. Ltd., Chennai, 2003.

Unit I	Chapter 5sections 5.1 to 5.4
Unit II	Chapter 5 sections 5.5 to 5.7
Unit III	Chapter 5 sections 5.8, Chapter VI sections 6.1 to 6.3
Unit IV	Chapter 7 sections 7.1 to 7.6
Unit V	Chapter 7 sections 7.7, 7.8 Chapter VIII sections 8.1, 8.2

Books for Reference:

1. S.Lang, Introduction to Linear Algebra 2nd Edition, Springer 2005.
2. AR.Vasistha, Modern Algebra, Krishna Prakashan Publication.

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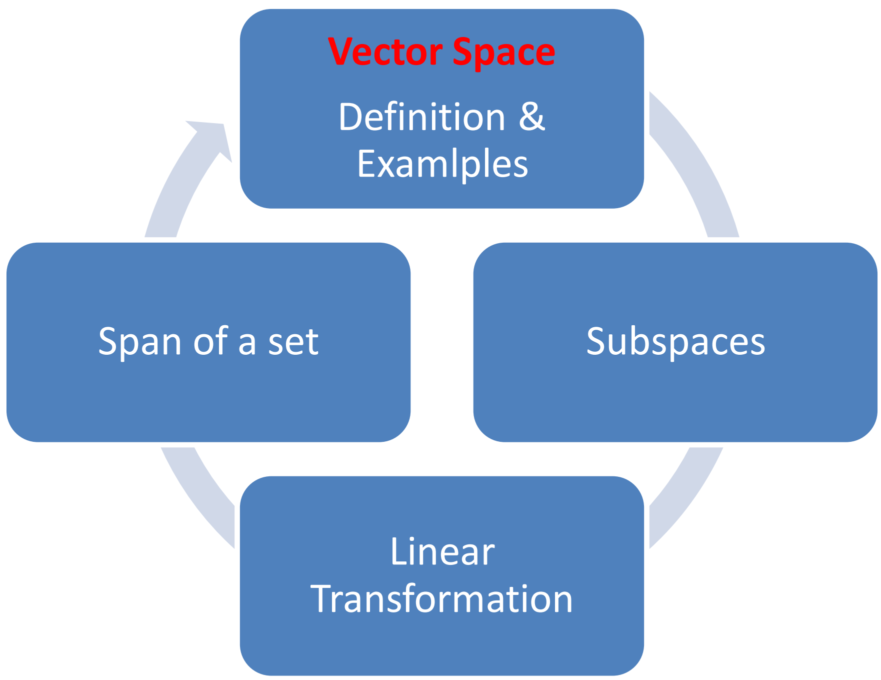
COURSE OUTCOME

Upon successful completion of this course students will be able to: 1) Use computational techniques and algebraic skills essential for the study of systems of linear equations, matrix algebra, vector spaces, Eigenvalues and Eigenvectors, orthogonality and diagonalization.

Unit I


Vector Space
Definition &
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CONTENT

VECTOR SPACE :

A non-empty set V is said to be a vector space over a field F if

(i) V is an abelian group under an operation called addition which we denote by +.
(ii) For every $\alpha \in F$ and $v \in V$, there is defined an element αv in V subject to the following conditions.

(a) $\alpha(u+v) = \alpha u + \alpha v$ for all $u,v \in V$ and $\alpha \in F$.
(b) $(\alpha+\beta)u = \alpha u + \beta u$ for all $u \in V$ and $\alpha , \beta \in F$.
(c) $\alpha(\beta u) = (\alpha\beta)u$ for all $u \in V$ and $\alpha , \beta \in F$.
(d) $1u = u$ for all $u \in V$.

Example:

R X R is a vector space over R under addition and scalar multiplication defined by,

$(x_1,x_2) + (y_1,y_2) = (x_1+y_1 , x_2 +y_2)$ and $\alpha(x_1,x_2) = (\alpha x_1 , \alpha x_2)$.

Theorem

Let V be a vector space over a field F. Then

(i) $\alpha 0=0$ for all $\alpha \in F$.
(ii) $0v = 0$ for all $v \in V$
(iii) $(-\alpha)v = \alpha(-v) = -(\alpha v)$ for all $\alpha \in F$ and $v \in V$.
(iv) $\alpha v = 0 \Rightarrow \alpha = 0$ or $v =0$.

Proof:

(i) $\alpha 0= \alpha (0+0) = \alpha 0 + \alpha 0$.
Hence $\alpha 0 = 0$.
(ii) $0v = (0+0)v = 0v + 0v$.
Hence $0v =0$.
(iii) $0 = [\alpha + (-\alpha)]v = \alpha v+(-\alpha)v$.
Hence $(-\alpha)v = -(\alpha v)$.
Similarly $\alpha(-v) = -(\alpha v)$.
Hence $(-\alpha)v = \alpha(-v) = -(\alpha v)$.
(iv) Let $\alpha v = 0$. If $\alpha = 0$, there is nothing to prove.
Therefore $\alpha \neq 0$.
Then $\alpha^{-1} \in F$.
Now , $v = 1v = (\alpha^{-1}\alpha)v = \alpha^{-1} (\alpha v) = \alpha^{-1} 0 = 0$.

Definition : Subspace

A subspace of a vector space V is a subset H of V which is itself a vector space with respect to the addition and scalar multiplication in V.

Example:

{0} and V are subspaces of any vector space V. They are called the trivial subspaces of V.

Theorem

Let V be a vector space over a field F. A non-empty subset W(V) is a subspace of V iff u,v $\in$ W and $\alpha,\beta \in F \Rightarrow \alpha u+\beta v \in W$.

Proof:

Let W be a subspace of V.

Let u,v $\in$ W and $\alpha,\beta \in F$.

Then αu & $\beta v \in W$ and hence $\alpha u+\beta v \in W$

conversely,

Let u,v $\in$ W and $\alpha,\beta \in F \Rightarrow \alpha u+\beta v \in W$.

Taking $\alpha = \beta = 1$, we get $u,v \in W \Rightarrow u+v \in W$.

Taking $\beta = 0$, we get $\alpha \in F$ & $u \in W \Rightarrow \alpha u \in W$.

Therefore W is a subspace of V.

Linear Transformations

Definition

A function L: V → W is linear if for all u, v ∈ V and r, s ∈ R we have

L(ru + sv) = rL(u) + sL(v).

Span of a set

Definition: Linear Combination

Let V be a vector space over F. Then, for any $\mathbf{u}_1, \ldots, \mathbf{u}_n \in$ V and $\alpha_1, \ldots, \alpha_n \in$ F, the vector $\alpha_1\mathbf{u}_1 + \ldots + \alpha_n\mathbf{u}_n = \sum_{i=1}^{n}\alpha_i\mathbf{u}_i$ is said to be a **linear combination** of the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

Definition: Linear Span

Let V be a vector space over F and $S \subseteq$ V. Then, the **linear span** of S, denoted $LS(S)$, is defined as LS(S) = $\alpha 1$ u1 +………+ αn un

That is, $LS(S)$ is the set of all possible linear combinations of finitely many vectors of S. If S is an empty set, we define $LS(S) = \{\mathbf{0}\}$

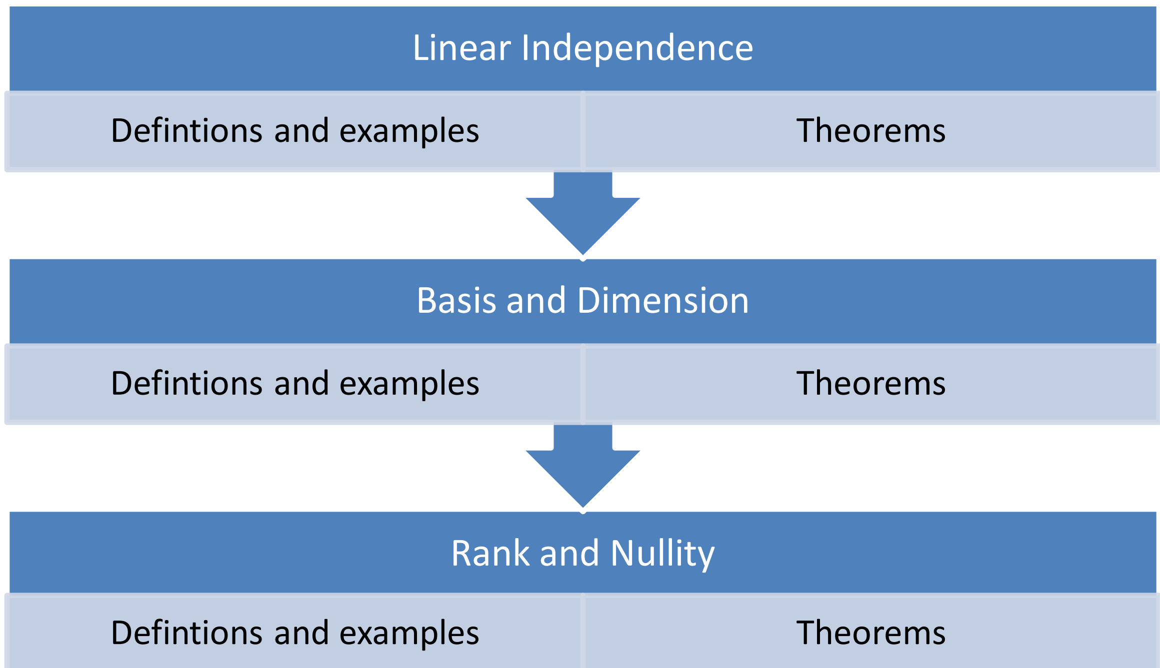
Summary

- Linear algebra is about linear combinations. That is, using arithmetic on columns of numbers called vectors and arrays of numbers called matrices, to create new columns and arrays of numbers.
- Linear algebra is the study of lines and planes, vector spaces and mappings that are required for linear transforms.
- In mathematics, physics, and engineering, a **vector space** (also called a **linear space**) is a set of objects called *vectors*, which may be added together and multiplied ("scaled") by numbers called *scalars*. Scalars are often real numbers, but some vector spaces have scalar multiplication by complex numbers or, generally, by a scalar from any mathematic field. The operations of vector addition and scalar multiplication must satisfy certain requirements, called vector *axioms* (listed below in § Notation and definition). To specify whether the scalars in a particular vector space are real numbers or complex numbers, the terms **real vector space** and **complex vector space** are often used.
- Linear transformations represented by 4 × 4 matrices are a fundamental operation in computer graphics and animation. Understanding their use, how to manipulate them, and how to control round-off error is an important first step in mastering graphics and animation techniques.

Questions:

1.Define a vector space.
2. Find L(S) in V_2 (R) if S={(1,0),(0,1)}.
3. Let V be a vector space over a field F.then prove that (i) $\alpha 0 = 0, \alpha \in \mathrm{F}, 0 \in \mathrm{V}$
(ii) $0v = 0, \alpha \in \mathrm{F}, \mathrm{v} \in \mathrm{V}$ (i) $(-\alpha)v = \alpha(-v) = (-\alpha v), \alpha \in \mathrm{F}, 0 \in \mathrm{V}$
4. If W={(a,b,c) R^3/lx+my+nz=0} Prove that W is a sub space.
5. Let V be a vector space and w be a subspace of V over f. define the quotient space V/w and prove that it is a vector space F, under the suitable operations in V/W.

Unit II


Linear Independence
Defintions and examples
Theorems
Basis and Dimension
Defintions and examples
Theorems
Rank and Nullity
Defintions and examples
Theorems

CONTENT

- Linear Independence
- Basis and Dimension
- Rank and Nullity.

Unit II

Linear Independence

Finite Dimensional Vector Space

Definition:

Let V be a vector space over F. Then, V is called **finite dimensional** if there exists $S \subseteq V$, such that S has finite number of elements and $V = LS(S)$. If such an S does not exist then V is called **infinite dimensional**.

Example

$\mathbb{R}^2$ is finite dimensional.

Theorem:

Let V be a vector space over F and $S \subseteq V$. Then, LS(S) is a subspace of V.

Proof:

By definition, $\mathbf{0} \in LS(S)$. So, $LS(S)$ is non-empty. Let $\mathbf{u},\mathbf{v} \in LS(S)$. To show, $a\mathbf{u} + b\mathbf{v} \in LS(S)$ for all $a,b \in F$. As $\mathbf{u},\mathbf{v} \in LS(S)$, there exist $n \in \mathbb{N}$, vectors $\mathbf{w}_i \in S$ and scalars $\alpha_i,\beta_i \in F$ suchthat $\mathbf{u} = \alpha_1\mathbf{w}_1 + \ldots\ldots + \alpha_n\mathbf{w}_n$ and $\mathbf{v} = \beta_1\mathbf{w}_1 + \ldots.. + \beta_n\mathbf{w}_n$. Hence,au +bv = $(a\alpha_1 + b\beta 1)W_1 + \ldots\ldots\ldots\ldots.. + (a\alpha_{Nn} + b\beta n)W_n \in LS(S)$.

Basis and Dimension

Definition:

Let W be a subspace of a vector space V. A set of vectors B = {v1, . . . , vk} in W is said to be a basis for W if (a) the set B spans all of W, that is, W = span{v1, . . . , vk}, and (b) the set B is linearly independent.

Definition:

Let V be a vector space. The dimension of V, denoted dim V, is the number of vectors in any basis of V. The dimension of the trivial vector space V = {0} is defined to be zero.

Note:
Let B = {v1, . . . , vn} be vectors in R n . If B is linearly independent then B is a basis for R n . Or if span{v1, v2, . . . , vn} = R n then B is a basis for R n .

Result:
Let V be a vector space. Then all bases of V have the same number of vectors.

Rank and Nullity:

Definition:

The rank of a matrix A is the dimension of its column space. We will use rank(A) to denote the rank of A.

Definition:

The nullity of a matrix A is the dimension of its nullspace Null(A). We will use nullity(A) to denote the nullity of A.

Theorem:

Let A be a $m \times n$ matrix. The rank of A is the number of leading 1's in its RREF. Moreover, the following equation holds: $n = \text{rank}(A) + \text{nullity}(A)$.

Proof:

A basis for the column space is obtained by computing and identifying the columns that contain a leading 1.
Each column of A corresponding to a column of with a leading 1 is a basis vector for the column space of A.
Therefore, if r is the number of leading 1's then $r = \text{rank}(A)$.
Now let $d = n - r$.
The number of free parameters in the solution set of $Ax = 0$ is d and therefore a basis for Null(A) will contain d vectors, that is, $\text{nullity}(A) = d$.
Therefore, $\text{nullity}(A) = n - \text{rank}(A)$.

Definition.

If the conditions in the above corollary are met, then T is called a nonsingular linear map. Otherwise, T is called singular. Notice that the terms singular and non-singular are only used for linear maps $T : U \to V$ for which U and V have the same dimension.

Summary

- In the theory of vector spaces, a set of vectors is said to be **linearly dependent** if there is a nontrivial linear combination of the vectors that equals the zero vector. If no such linear combination exists, then the vectors are said to be **linearly independent**. These concepts are central to the definition of dimension.

- Vector spaces are the subject of linear algebra and are well characterized by their dimension, which, roughly speaking, specifies the number of independent directions in the space.

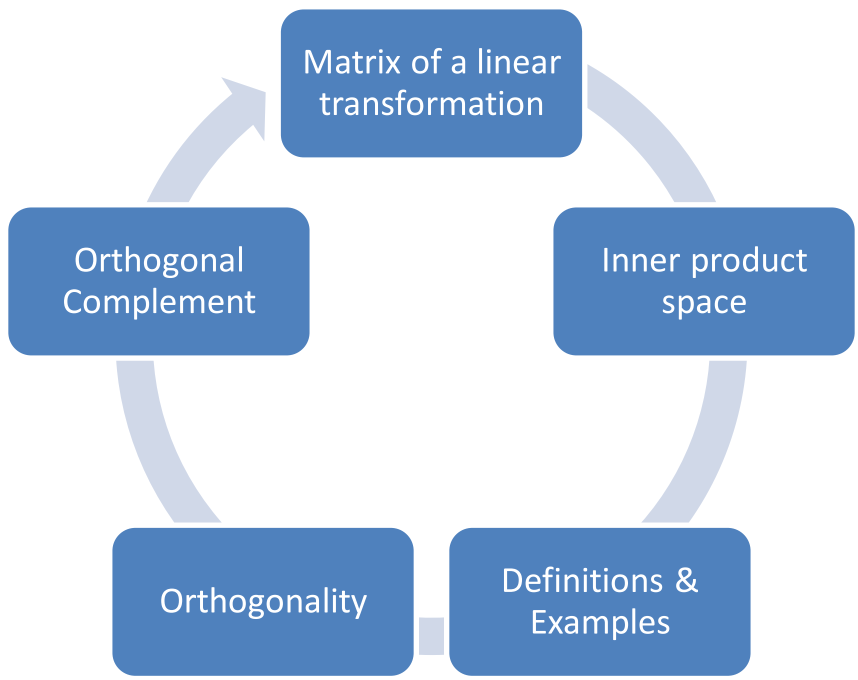
- The **rank–nullity theorem** is a theorem in linear algebra, which asserts that the dimension of the domain of a linear map is the sum of its rank (the dimension of its image) and its *nullity* (the dimension of its kernel).

Questions:

1. Define dimension of a vector space v over F.
2.Test whether (1,0,0),(0,1,0),(0,0,1) are linearly independent or not
3. Prove that any two basis of finite dimension vector space have the same number of elements
4.Show that any n vector space of dimension over a field F is isomorphic to $V_n(F)$.

5.Let V be a finite dimension vector space and W be a subspace of V, then prove that dim(V/W) = dimV- dim W.

Unit III


Matrix of a linear transformation
Inner product space
Definitions & Examples
Orthogonality
Orthogonal Complement

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- Matrix of a Linear Transformation
- Inner Product Space – Definition and examples
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- Orthogonal complement.

Unit III

Matrix of a linear transformation

Let V and W be vector spaces, with bases S = {e1, . . . , en} and T = {f1, . . . , fm} respectively. Since these are bases, there exist constants v i and w j such that any vectors v ∈ V and w ∈ W can be written as: $v = v_1 e_1 + v_2 e_2 + \cdot\cdot\cdot + v_n e_n$

$w = w_1 f_1 + w_2 f_2 + \cdot\cdot\cdot + w_m f_m$

Inner Product Space:

An inner product is a generalization of the dot product. In a vector space, it is a way to multiply vectors together, with the result of this multiplication being a scalar.

More precisely, for a real vector space, an inner product $\langle \cdot, \cdot \rangle$ satisfies the following four properties. Let u, v, and w be vectors and α be a scalar, then:

1. $.<u+v,w> = <u.w>+<v,w>$

2. $. <\alpha v,w>= \alpha<v,w>$

3. $.<v,w>=<w,v>$

4. $<v,v>=0$ and equal if and only if v=0.

Examples of inner product spaces include:

1. The real numbers $\mathbb{R}$, where the inner product is given by

$<x,y> = xy$ (1)

2. The Euclidean space $\mathbb{R}^n$, where the inner product is given by the dot product

$<(x_1,x_2,\ldots xn),(y_1,y_2,\ldots,yn)> = x_1y_1+x_2y_2+\ldots+xnyn$ (2)

3. The vector space of real functions whose domain is an closed interval with inner product

$$\langle f, g \rangle = \int_a^b f\, g\, d\,x.$$

Property:

Let V be a vector space and **u**, **v**, and **w** be vectors in V and c be a constant. Then an *inner product* (,) on V is a function with domain consisting of pairs of vectors and range real numbers satisfying the following properties.

1. (**u**, **u**) $\geq$ 0 with equality if and only if **u** = **0**.

2. (**u**, **v**) = (**v**, **u**)

3. (**u** + **v**, **w**) = (**u**, **w**) + (**v**, **w**)

4. (c**u**, **v**) = (**u**, c**v**) = c(**u**, **v**)

A vector space with its inner product is called an *inner product space*.

Orthogonal :

For any inner product space V we call vectors **v** and **w** *orthogonal* if

(**v**, **w**) = 0

Length:

And we define the *length* of **v** by

Length + $\sqrt{(v, v)}$

Norm

The notion of norm generalizes the notion of length of a vector in R n . Definition. Let V be a vector space. A function α : V → R is called a norm on V if it has the following properties:

(i) α(x) ≥ 0, α(x) = 0 only for x = 0 (positivity)

(ii) α(rx) = |r| α(x) for all r ∈ R (homogeneity)

(iii) α(x + y) ≤ α(x) + α(y) (triangle inequality)

Normed vector space:

Definition.

A normed vector space is a vector space endowed with a norm. The norm defines a distance function on the normed vector space: dist(x, y) = kx − yk. Then we say that a sequence x1, x2, . . .

converges to a vector x if dist(x, xn) → 0 as n → ∞. Also, we say that a vector x is a good approximation of a vector x0 if dist(x, x0) is small.

Definitions:

Orthogonal

- In geometry, two Euclidean vectors are **orthogonal** if they are perpendicular, *i.e.*, they form a right angle.
- Two vectors, *x* and *y*, in an inner product space, *V*, are *orthogonal* if their inner product <x,y>= 0. This relationship is denoted x $\perp$ Y.

orthonormal

- An orthogonal matrix is a matrix whose column vectors are orthonormal to each other.

orthogonal subspaces

- Two vector subspaces, *A* and *B*, of an inner product space *V*, are called **orthogonal subspaces** if each vector in *A* is orthogonal to each vector in *B*.

Orthogonal complement.

The largest subspace of *V* that is orthogonal to a given subspace is its orthogonal complement.

Summary

- Linear transformation: Let V and W be two vector spaces over R, the field of real numbers. A mapping T from V to W is called a linear transformation from V to W if (i) $T(u + v) = T(u) + T(v)$ for every pair of vectors u and v from V and (ii) $T(\alpha v) = \alpha T(v)$, for every vector v from V and scalar α from R..

- Inner product space, In mathematics, a vector space or function space in which an operation for combining two vectors or functions (whose result is called an inner product) is defined and has certain properties.

Questions:

1.If nullity T =0 then rank T= ?

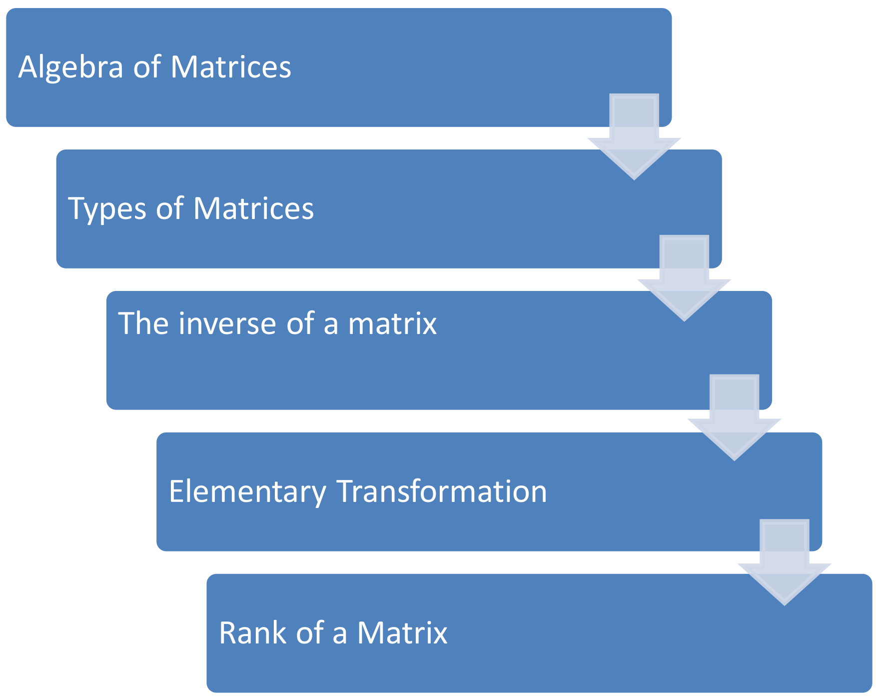
2.Find the kernel of T: V_3(R)--- V_3(R) defined T(a,b,c)=(a,b,0)

3.Find the linear transformation T: V_3(R)--- V_3(R) determined by the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$ with respect to the standard basis.

4.Show that the set $M_{mxn}(F)$ of all mX n matrix over a field Fis an vector space of dimension mn over F.

5.Let V be a vector space and A and B be two subspace of v over F, then prove that $\frac{A+B}{B} = \frac{A}{A \cap B}$

Unit IV


Algebra of Matrices
Types of Matrices
The inverse of a matrix
Elementary Transformation
Rank of a Matrix

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- Algebra of Matrices
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- Elementary Transformations
- Rank of a Matrix
- Simultaneous linear equations.

Algebra of Matrices:

Definition : Square Matrices:

If mxn matrix A is an array of mn numbers aij , where 1≤i≤m, 1≤j≤n, arranged in m rows and n columns as follows:

$$\begin{bmatrix} a11 & \cdots & a1n \\ \vdots & \ddots & \vdots \\ am1 & \cdots & amn \end{bmatrix}$$

We shall denote this matrix by the symbol (aij). If m=n, A is called a square matrix of order n.

Definition : Equal Matrices:

Two matrices A=(aij) and B=(bij) are said to be equal if A & B have the same number of rows and colums and the corresponding entries in the two matrices are same.

Definition : Addition of Matrices:

Two matrices must have an equal number of rows and columns to be added.If

A=(aij) and B=(bij) by A+B = (aij +bij).

Example :

If A = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, B = $\begin{pmatrix} 1 & 3 \\ 4 & 4 \end{pmatrix}$ then A + B = $\begin{pmatrix} 2 & 5 \\ 7 & 8 \end{pmatrix}$

Definition : Product of Matrices:

If **A** is an $m \times n$ matrix and **B** is an $n \times p$ matrix, the *matrix product* **C** = **AB** (denoted without multiplication signs or dots) is defined to be the $m \times p$ matrix.

Definition : Transpose of a Matrix:

The transpose of a matrix is **found by interchanging its rows into columns or columns into rows**. The transpose of the matrix is denoted by using the letter “T” in the superscript of the given matrix.

For example,

if “A” is the given matrix, then the transpose of the matrix is represented by A' or A^T.

Definition : conjugate matrix

A conjugate matrix is a matrix obtained from a given matrix by taking the complex conjugate of each element .

Definition : complex conjugate transpose

The complex conjugate transpose of a matrix interchanges the row and column index for each element, reflecting the elements across the main diagonal. The operation also negates the imaginary part of any complex numbers. For example, if **B = A'** and A(1,2) is 1+1i , then the element B(2,1) is 1-1i .

Types of Matirces:

1. Column Matrix
2. Diagonal Matrix
3. Scalar Matrix
4. Upper triangular Matrix
5. Lower triangular Matrix

Definition : symmetric

If the transpose of a matrix is equal to itself, that matrix is said to be symmetric. Two examples of symmetric matrices appear below. Note that each of these matrices satisfy the defining requirement of a symmetric matrix:
A = A' and B = B'.

Definition : skew symmetric

A skew symmetric matrix is defined as the square matrix that is equal to the negative of its transpose matrix. For any square matrix, A, the transpose matrix is given as A^T. A skew-symmetric or antisymmetric matrix A can therefore be represented as, $A = -A^T$

Prove that A+B is skew-symmetric.
We have

$$(A+B)^T=A^T+B^T=(-A)+(-B)=-(A+B).$$

Definition : unitary matrix

A unitary matrix is a matrix whose inverse equals it conjugate transpose. Unitary matrices are the complex analog of real orthogonal matrices. The conjugate transpose U* of U is unitary.

Inverse of a Matrix:

Definition : Inverse matrix

If A is a non-singular square matrix, there is an existence of n x n matrix A^{-1}, which is called the inverse matrix of A such that it satisfies the property:

$AA^{-1} = A^{-1}A = I$, where I is the Identity matrix

The identity matrix for the 2 x 2 matrix is given by

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Definition : singular matrix

A matrix is said to be singular if and only if its determinant is equal to zero. A singular matrix is a matrix that has no inverse such that it has no multiplicative inverse.

Definition : Non Singular matrix

Non Singular matrix is a square matrix whose determinant is a non-zero value.

Definition : Adjoint of a matrix

Let A=[aij] be a square matrix of order n . The adjoint of a matrix A is the transpose of the cofactor matrix of A . It is denoted by adj A . An adjoint matrix is also called an adjugate matrix.

Definition : Invertible matrix

An invertible matrix is a square matrix that has an inverse. We say that a square matrix is invertible if and only if the determinant is not equal to zero. In other words, a 2 x 2 matrix is only invertible if the determinant of the matrix is not 0.

Summary

- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns. The dimensions of a matrix refer to the number of rows and the number of columns. A 3×2 3×2 matrix has three rows and two columns.
- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously. The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently.

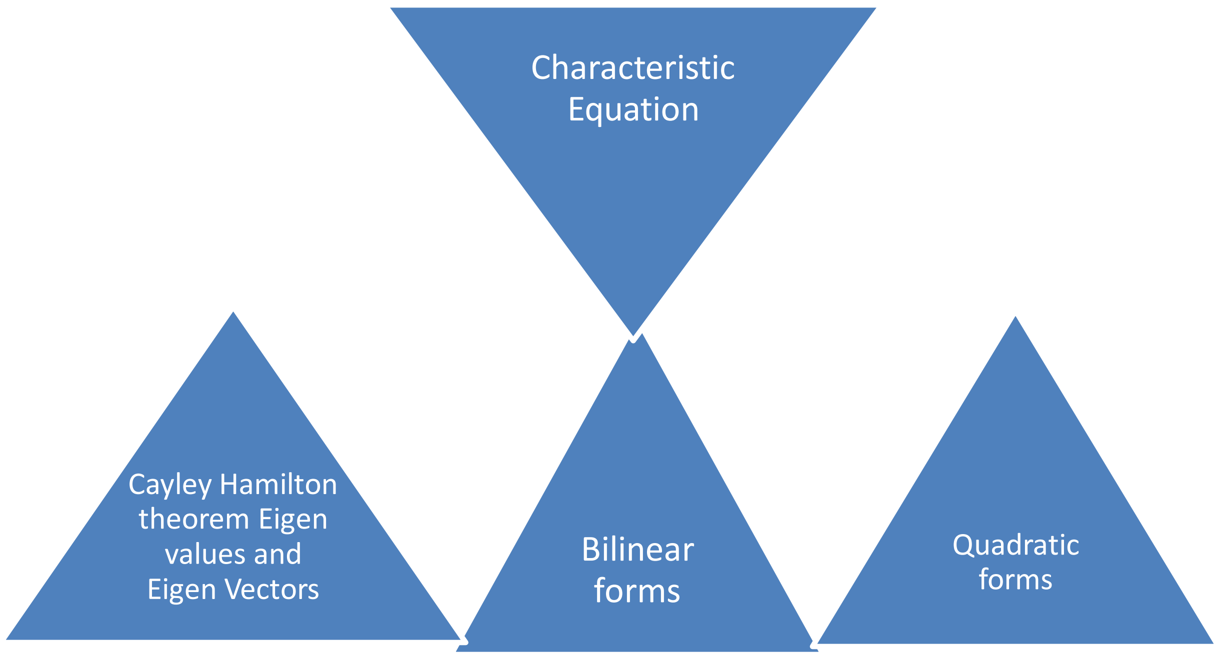
Questions:

1.Define inner product space.
2. Find the value of $(u, \alpha v)$
3.Solve x+2y-5z= -9, 3x-y+2z = 5, 2x+3y-z=3

4.Prove that any square matrix A can be expressed uniquely as the sum of a symmetric and skew symmetric

5.a) State and prove gram schmidth orthogonalisation process
b) Construct an ortho normal basis for $V_3(R)$ with respect to the standard inner product space for the basis $\{V_1=(1,0,1), V_2=(1,3,1), V_3=(3,2,1)$

UNIT-V


Characteristic
Equation
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Quadratic
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- Characteristic Equation and
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- Eigen values and Eigen Vectors,
- Bilinear forms –
- Quadratic forms.

Characteristic Equation

Definition : Characteristic Equation:

The characteristic equation is the equation which is solved to find a matrix's eigenvalues, also called the characteristic polynomial. For a general $k \times k$ matrix A, the characteristic equation in variable λ is defined by

$$\det(\mathsf{A} - \lambda\, \mathsf{I}) = 0, \quad (1)$$

where I is the identity matrix and $\det(\mathsf{B})$ is the determinant of the matrix B. Writing A out explicitly gives

$$\mathsf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}, \quad (2)$$

so the characteristic equation is given by

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} - \lambda \end{vmatrix} = 0$$

Ie, The characteristic equation, also known as the determinantal equation, is **the equation obtained by equating the characteristic polynomial to zero**.

Theorem : Cayley Hamilton Theorem

Every square matrix over a commutative ring (such as the real or complex field) satisfies its own characteristic equation.

A matrix satisfies its own characteristic equation. That is, if the characteristic equation of an $n \times n$ matrix **A** *is* $\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0$, then An+an−1An−1+⋯+a1A+a0I=0.

Note once again that when we change a scalar equation to a matrix equation, the unity element 1 is replaced by the identity matrix **I.**

Eigen Values And Eigen Vectors:

Suppose, $A_{n\times n}$ is a square matrix, then $[A- \lambda I]$ is called an Eigen or characteristic matrix, which is an indefinite or undefined scalar. Where determinant of Eigen matrix can be written as, **$|A- \lambda I|$ and $|A- \lambda I| = 0$** is the Eigen equation or characteristics equation, where "I" is the identity matrix. The roots of an Eigen matrix are called Eigen roots..

Properties of Eigen values

- Eigenvectors with Distinct Eigenvalues are Linearly Independent
- Singular Matrices have Zero Eigenvalues
- If A is a square matrix, then $\lambda = 0$ is not an eigenvalue of A
- **For a scalar multiple of a matrix:** If A is a square matrix and λ is an eigenvalue of A. Then, $a\lambda$ is an eigenvalue of aA.
- **For Matrix powers:** If A is square matrix and λ is an eigenvalue of A and $n\geq 0$ is an integer, then λ^n is an eigenvalue of A^n.
- **For polynomials of matrix:** If A is a square matrix, λ is an eigenvalue of A and p(x) is a polynomial in variable x, then $p(\lambda)$ is the eigenvalue of matrix p(A).
- **Inverse Matrix:** If A is a square matrix, λ is an eigenvalue of A, then λ^{-1} is an eigenvalue of A^{-1}
- **Transpose matrix:** If A is a square matrix, λ is an eigenvalue of A, then λ is an eigenvalue of A^t

Problems :

Find the characteristic roots of the matrix $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$

Solution:

The characteristic equation of A is given by $|A-\lambda I|=0$.

Ie, $\begin{vmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{vmatrix} = 0$,

$(2-\lambda)(1-\lambda) - 12 = 0$,

$2-2\lambda - \lambda + \lambda^2 - 12 = 0$,

$\lambda^2 - 3\lambda - 10 = 0$,

$\lambda = 5, -2$.

Definition : Bilinear form

Bilinear form A bilinear form on a real vector space V is a function f : V × V → R which assigns a number to each pair of elements of V in such a way that f is linear in each variable.

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Definition : Matrix of Bilinear form

Matrix of a bilinear form Suppose that h,i is a bilinear form on V and let v1, v2, . . . , vn be a basis of V . The matrix of the form with respect to this basis is the matrix A whose entries are given by aij = hvi , vj i for all i, j.

Definition :Positive definite

A bilinear form h,i on a real vector space V is positive definite, if hv, vi > 0 for all v 6= 0. A real n × n matrix A is positive definite, if x tAx > 0 for all x 6= 0.

Definition : Symmetric Bilinear form

Symmetric A bilinear form h,i on a real vector space V is called symmetric, if hv, wi = hw, vi for all v, w ∈ V . A real square matrix A is called symmetric, if aij = aji for all i, j.

Definition: Quadratic Form

Let f be a symmetric bilinear form defined by V. The the quadratic form associated with f is the mapping q:V-F defined by q(v) = f(v,v).
The matrix of the bilinear form f is called the matrix of the associated quadratic form q.

Summary

- Characteristic equation, the equation obtained by equating to zero the characteristic polynomial of a matrix or of a linear mapping. Method of characteristics, a technique for solving partial differential equations.
- In mathematics, a bilinear form on a vector space V (the elements of which are called vectors) over a field K (the elements of which are called scalars) is a bilinear map $V \times V \to K$.
- Quadratic equations are equations of the form $y = ax^2 + bx + c$ or $y = a(x - h)^2 + k$. The shape of the graph of a quadratic equation is a parabola.

Questions

1. Define minimal generating set.

2. Find the rank of the matrix $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 6 & -1 & 1 \end{pmatrix}$

3. State and prove cayley Hamilton theorem..

4. Find the eigen values and eigen vectors of matrix A= $\begin{pmatrix} 0 & 1 & 1 \\ -4 & 4 & 2 \\ 4 & -3 & -1 \end{pmatrix}$

5.Prove that Similar matrixes have the same characteristic roots