Dr.UmayalRamanathan College for Women, Karaikudi – 3
Accredited with B+ Grade by NAAC
Affiliated to Alagappa University
(Run by Dr.AlagappaChettiar Educational Trust)
Department of Mathematics

TOPOLOGY –I

7MMA3C2

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SYLLABUS
COURSE CODE: 7MMA3C2
TOPOLOGY – I

Unit I

Topological Spaces – Basis of a topology – the order topology – the product topology on XxY– the subspace topology – closed sets and limit points.

Unit II

Continuous functions – the product topology – the metric topology – the quotient topology.

Unit III

Connected spaces – connected sets in the real line – components and path components – local connectedness.

Unit IV

Compact spaces – compact sets in the real line – limit point compactness.

Unit V

The countability axioms – the separation axioms – the Urysohn's lemma – the Uryshon'smetrization theorem.

Text Book

James R.Munkres, Topology a first course, Prentice Hall of India Pvt. Ltd.,New Delhi (1987)

Chapter II	:	(Sections 2.1 to 2.10)
Chapter III	:	(Sections 3.1 to 3.4)
Chapter IV	:	(Sections 3.5 to 3.7)
Chapter V	:	(Sections 4.1 to 4.4)

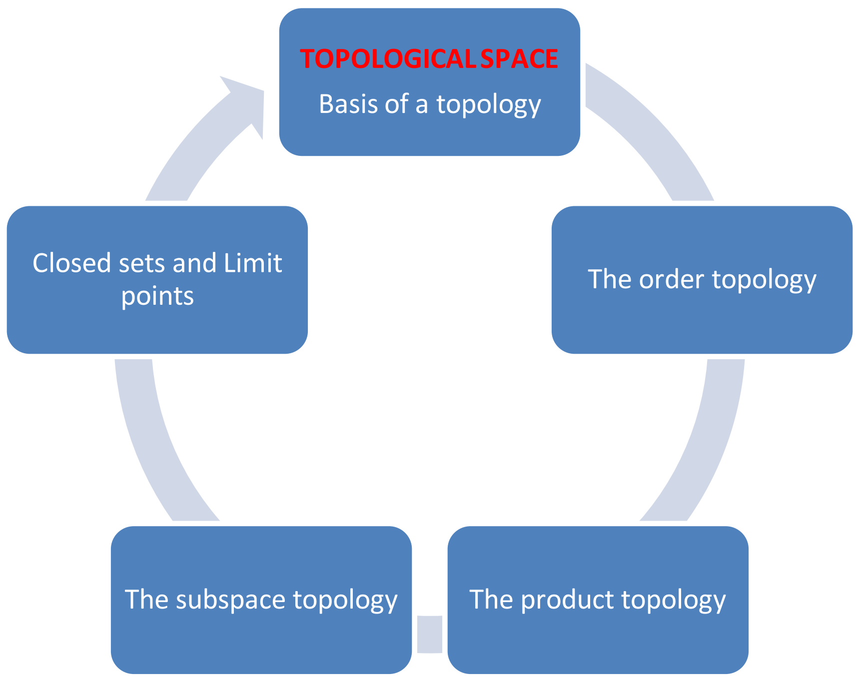
Books for Supplementary Reading and Reference:

1. James Dugundji, Topology, Prentice Hall of India, New Delhi, 1975.
2. George F.Simmons, Introduction to Topology and Modern Analysis, McGraw Hill Book Co., 1963.

COURSE OUTCOME

III	Topology – I	7MMA3C2	CO1	They get the knowledge Topological Spaces, Basis of a topology , the order topology , the product topology on XxY, the subspace topology, closed sets and limit points.
			CO2	To know about the concept of Continuous functions, the product topology , the metric topology , the quotient topology.
			CO3	They can understandConnected spaces , connected sets in the real line , components and path components , local connectedness.
			CO4	They get the knowledge Compact spaces , compact sets in the real line – limit point compactness. .
			CO5	Understand the The countability axioms , the separation axioms
			CO6	To know the Urysohn's lemma , the Uryshon's metrization theorem.

Unit I


TOPOLOGICAL SPACE
Basis of a topology
The order topology
The product topology
The subspace topology
Closed sets and Limit points

Content

- Topological Spaces
- Finite Complement Topology
- Union and Intersection of topologies
- Basis of a topology
- The order topology
- The product topology on XxY
- The subspace topology
- Closed sets and limit points.

TOPOLOGICAL SPACE

TOPOLOGY:

Let X be a non-empty set. Let τbe a collection of subset of X.

$\tau$is said to be topology on X.Ifτsatisfies the followingcondition.

1. $\emptyset$,X $\in\tau$
2. $A_1, A_2 \ldots\ldots\ldots A_n \in \tau \Longrightarrow \bigcap_{i=1}^{n} A_i \in \tau$
3. $\forall_\alpha \in \tau, A_\alpha \in \tau \Longrightarrow \bigcup_{\alpha\in J} A_\alpha \in \tau$

The pair (X,τ) is called a topological space .

Note:1

The members of τ are called the open sets in X (or) τ open set in X.

Note:2

IF (X , τ) is topological space then the element of X are called points.

Example:1

Let X be any set and τbe the collection of all subsets of X.Thenτ is the topology on X . This topology $\tau$is called discrete topology.

Example:2

Trivial topology (or) in discrete topology .The collection consisting of X and $\emptyset$only is also a topology on X.

Then it is said to be trivial topology (or) indiscrete topology.

Example:3

Let X = { a , b , c }

C is the collection of subset of X then

C = { {a} . {b} . {c} . {a,b} . {b,c} ,{c,a}, X ,$\emptyset$}

τ = { {a},{a,b}, X ,$\emptyset$}

Therefore , τ satisfies the topology.

FINITE COMPLEMENT TOPOLOGY :

X be a set.Letτ_F be the collection of subsets of U(X) such that X – U either is finite (or) is all of X then τ_F is a topology on X called the finite complement topology.

RESULT:1

The intersection of two topology is again a topology.

Let τ_1 and τ_2 be the two topology on X

To prove:

τ_1 and τ_2 is a topology on X

1. $\emptyset \in \tau_1$, $\emptyset \in \tau_2$

$\Longrightarrow \emptyset \in \tau_1 \cap \tau_2$

$X \in \tau_1$, $X \in \tau_2$

$\Longrightarrow X \in \tau_1 \cap \tau_2$
$\Longrightarrow \emptyset , X \in \tau_1 \cap \tau_2$

2. Let $A_1, A_2 \ldots\ldots\ldots. An \in \tau_1 \cap \tau_2$

$\Longrightarrow A_1, A_2 \ldots\ldots\ldots.A_n \in \tau_1$ and $\bigcap_{i=1}^{n} A_i \in \tau_1$

Similarly , $\bigcap_{i=1}^{n} A_i \in \tau_2$

$\Longrightarrow \bigcap_{i=1}^{n} A_i \in \tau_1 \cap \tau_2$

3. $A_\alpha \in \tau_1 \cap \tau_2$

$\Longrightarrow A_\alpha \in \tau_1 \& \alpha \in \tau$

$\Longrightarrow \bigcup_{\alpha \in J} A_\alpha \in \tau_1$

$\Longrightarrow A_\alpha \in \tau \& \alpha \in \tau$

$\Longrightarrow \bigcup_{\alpha \in J} A_\alpha \in \tau_2$

$\Longrightarrow \bigcup_{\alpha \in J} A_\alpha \in \tau_1 \cap \tau_2$

$\therefore \tau_1 \cap \tau_2$ is a topology on X.

RESULT : 2

The union of the topology need not be the topology

Let $X = \{a,b,c\}$

$\tau_1 = \{\emptyset , X , \{a\}\}$

$\tau_2 = \{\emptyset , X , \{b\}\}$

Then τ_1 and τ_2 are topology on x.

Now,

$\tau_1 \cup \tau_2 = \{\emptyset , X , \{b\}\}$

$\{a\} \in \tau_1 \cup \tau_2$

$\{b\} \in \tau_1 \cup \tau_2$

$\{a\} \cup \{b\} = \{a,b\} \notin \tau_1 \cup \tau_2$

$\therefore \tau_1 \cup \tau_2$ is not a topology on X

$\therefore$The union of the topology need not be the topology.

DEFINITION:

Let X have two topology $\boldsymbol{\tau}$&$\boldsymbol{\tau}$' .We say that $\boldsymbol{\tau}$ is finer than $\boldsymbol{\tau}$'($\boldsymbol{\tau} \supset \boldsymbol{\tau}$')

Basis for a topology

Let X be a non empty set .Let B be a collection of subset of X

B is called a basis if its satisfies the following condition

I. $\forall x \in X$, $\exists B \in \mathbb{B} \ni : x \in B$
II. $\forall\ x \in B_1 \cap B_2$ Where $B_1, B_2 \in \mathbb{B}$

$\exists\ B_3 \in \mathbb{B} \ni :\ x \in B_3 \subset\ B_1 \cap B_2$

Note:

The element in a basis are called basis element .The element in a topology are called open sets.

Let X be a non empty set . Let $\mathbb{B}$ be a basis of X.

Thus topology $\boldsymbol{\tau}$ generated by the basis $\mathbb{B}$ is define as follows,

U is open in X$\Leftrightarrow \forall x \in U$ [(i.e)$U \in \tau$]there exists $B \in \mathbb{B} \ni: x \in B \subset U$

NOTE :

Every basis element is an open set

LEMMA :

Let X be a non empty set .let ∧ topology$\boldsymbol{\tau}$ on X .Then $\boldsymbol{\tau}$'is equal to collection of all union of elements of $\mathbb{B}$.

Given is a basis for the topology $\boldsymbol{\tau}$ on X

Let$\boldsymbol{\tau}$' be the collection of all union of elements of $\mathbb{B}$

To prove :

$\boldsymbol{\tau}$ is equal to $\boldsymbol{\tau}$'

(i.e) to prove

I. $\boldsymbol{\tau} \subset \boldsymbol{\tau}'$
II. $\boldsymbol{\tau}' \subset \boldsymbol{\tau}$

$\boldsymbol{\tau} \subset \boldsymbol{\tau}'$

Let U $\in \boldsymbol{\tau} \Rightarrow$ U is open in X (by definition)

$\Rightarrow \forall x \in U \exists B_x \in \mathbb{B}$

$\Rightarrow x \in B_x \subset U$ (by definition)

$\Rightarrow U = \bigcup_{x \in U} B_x$

$\Rightarrow$ U is the union of basis of elements

$\Rightarrow U \in \boldsymbol{\tau}'$

$\therefore \boldsymbol{\tau} \subset \boldsymbol{\tau}'$……….(1)

$\boldsymbol{\tau}' \subset \boldsymbol{\tau}$

Let $v \in \boldsymbol{\tau}' \Rightarrow v = \bigcup_{\alpha \in \boldsymbol{\tau}} B_\alpha$ where $B_\alpha \in \mathbb{B}$

$\Rightarrow B_\alpha$ is a basis element

$\Rightarrow B_\alpha$ is open in X [member of $\boldsymbol{\tau}$ is open in X]

$\Rightarrow B_\alpha \in \boldsymbol{\tau}$

But $\boldsymbol{\tau}$ is a topology

$\Rightarrow \bigcup_{\alpha \in \boldsymbol{\tau}} B_\alpha \in \boldsymbol{\tau}$

$\Rightarrow v \in \boldsymbol{\tau}$

$\therefore \boldsymbol{\tau}' \subset \boldsymbol{\tau}$ ……….(2)

$\therefore$From (1) & (2)

$\boldsymbol{\tau} = \boldsymbol{\tau}'$

LEMMA :

Let $\mathbb{B}$ and $\mathbb{B}'$ be a basis for the topology $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ respectively on X. Then the following are equivalent.

i. $\boldsymbol{\tau}'$ is finer than $\boldsymbol{\tau}$
ii. For each $x \in X$ and each basis element $B \in \mathbb{B} \subset x$ there is a basis element $B' \subset \mathbb{B}' \ni : x \in B' \subset \mathbb{B}$

(i)$\Longrightarrow$(ii)

Assume that

$\boldsymbol{\tau}$’ is finer than $\boldsymbol{\tau}$

(i.e)$\boldsymbol{\tau}' \supset \boldsymbol{\tau}$ or $\boldsymbol{\tau} \subset \boldsymbol{\tau}'$

Let $x \in X$ and $B \in \mathbb{B}$ be a basis element and $x \in B$

$B \in \mathbb{B} \Longrightarrow$ B is open in X

$\Longrightarrow B \in \boldsymbol{\tau}$

$\Longrightarrow B \in \boldsymbol{\tau}'$ [since $\boldsymbol{\tau} \subset \boldsymbol{\tau}'$]

$\Longrightarrow B \in \mathbb{B}'$ $\forall x \in B$

$\Longrightarrow$ There exists $B' \in \mathbb{B}'$ such that $x \in B' \Longrightarrow x \in B' \subset B$

Hence (i)$\Longrightarrow$(ii)

(ii)$\Longrightarrow$(i)

$x \in X$ and $B \in \mathbb{B} \subset X$

(ie) $\boldsymbol{\tau} \subset \boldsymbol{\tau}'$

Let $U \in \boldsymbol{\tau}$

$\Longrightarrow$ U is open in X

$\Longrightarrow \forall x \in U$

$\Longrightarrow B \in \mathbb{B}$

Such that $x \in B \subset U$

$X \in U \Longrightarrow x \in X$. [$\therefore U(x)$]

By hypothesis,

There exists$B' \in \mathbb{B}'$

Such that $x \in B' \subset B$

(ie) $x \in B' \subset B \subset U$

$\Longrightarrow$U is open

$\Rightarrow U \in \boldsymbol{\tau}'$

$\therefore \boldsymbol{\tau} \subset \boldsymbol{\tau}'$

Hence $\boldsymbol{\tau}'$ is finer than $\boldsymbol{\tau}$.

SUMMARY

- Topology is a relatively new branch of mathematics; most of the research in topology has been done since 1900.
- Topology studies properties of spaces that are invariant under any continuous deformation.
- Topological concepts and methods underlie much of modern mathematics.
- Topological approach has clarified very basic structural concepts in many of its branches.

QUESTIONS

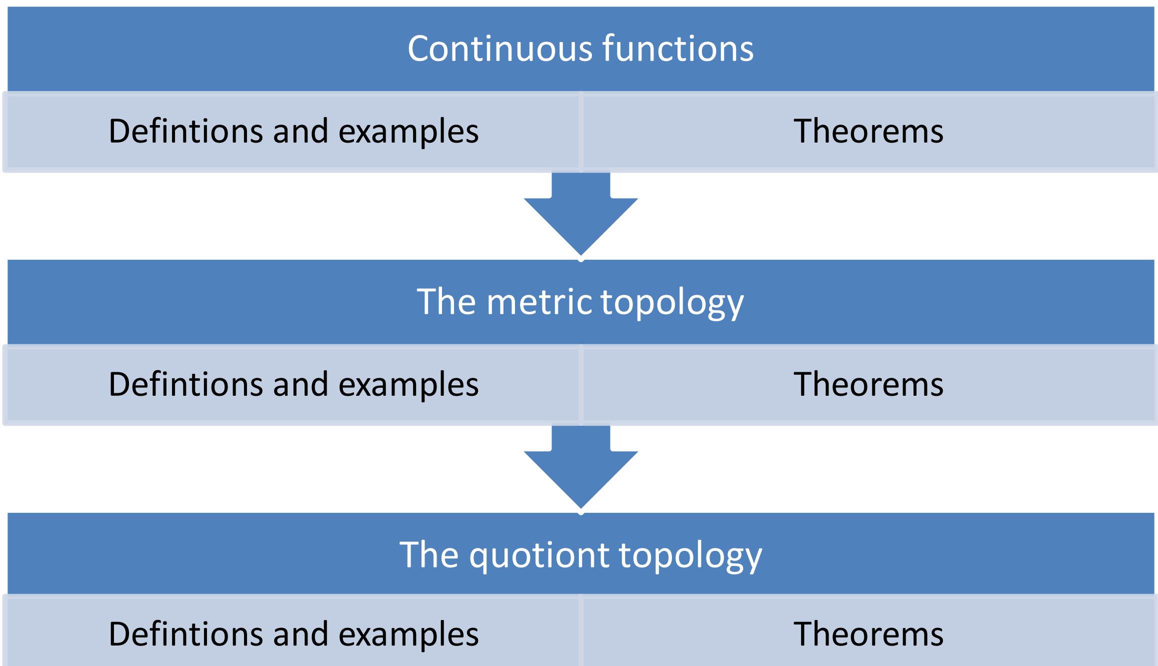
1. Define Topological Space.
2. Explain the example of a topological space.
3. Define Metrizable Space.
4. Define Quotient topology.
5. Define linear continuum.
6. Define Path connected.
7. Prove that the rational number Q are not connect5. Let $\mathbb{B}$ and $\mathbb{B}'$ be a basis for the topology $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$ respectively on X. Then the following are equivalent.

   (a)$\boldsymbol{\tau}'$ is finer than $\boldsymbol{\tau}$ (b)For each $x \in X$ and each basis element $B \in \mathbb{B} \subset x$ there is a basis element

   $B' \subset \mathbb{B}' \ni : x \in B' \subset \mathbb{B}$.

8. Prove that X be a non empty set .let $\wedge$ topology$\boldsymbol{\tau}$ on X .Then $\boldsymbol{\tau}'$is equal to collection of all union of elements of $\mathbb{B}$**.**
9. The topologies on R^n induced by the Euclidean metric d and the square metric R are the same as the product topology on $R^{n.}$That is R^n is Metrizable.
10. State and prove sequence lemma.

Unit II


Continuous functions
Defintions and examples
Theorems
The metric topology
Defintions and examples
Theorems
The quotiont topology
Defintions and examples
Theorems

Content:

- Continuous functions
- The product topology
- The metric topology
- The quotient topology.

CONTINUOUS FUNCTION:

Let X and Y are topological spaces. A function f : X→Y is said to be Continuous if for each open subset V(Y),the set f^{-1}(v) is an open subset of X.

THEOREM:

Let X and Y be topological space. Let f : X →Y then the following are equivalent

(i) f is continuous.
(ii) For every closed set B of X the set f^{-1} (B) is closed in X.
(iii) For every subset A of X, one has f($\bar{A}$) ⊂ $\overline{F(A)}$.
(iv) For each $x \in X$ and each neighbourhood ∪ of X such that f(u)=v.

(i)⟹(ii)

Assume that,

f is continuous.

To prove:

$f\overline{(A)} \subset f\overline{(A)}$∀ A is a subset of X.

Let A ⊂ X

Let $y \in f\overline{(A)}$

$\Rightarrow y = f(x)$, $x \in \bar{A}$

$\Longrightarrow$ $\forall$open set containing x intersect A.

To prove:

$$y \in \overline{f(A)}$$

Let v be a neighbourhood of y that intersect f(A).

i.e) $y \in v$

$$\Longrightarrow y \in v$$

$$\Longrightarrow f(x) \in v$$

$$\Longrightarrow x \in f^{-1}(v)$$

$$\left.\begin{matrix} f: X \to Y \;\; is\ continuous \\ V\ is\ open\ in\ Y \end{matrix}\right\} \Longrightarrow f^{-1}(v) \text{is open in x.}$$

$\Longrightarrow f^{-1}(v)$is open in x.

But, $x \in \bar{A} \Longrightarrow f^{-1}(v) \cap A \neq \varphi$

Let $a \in f^{-1}(v) \cap A$

$$\Longrightarrow a \in f^{-1}(v) \& \; a \in A$$

$\Rightarrow f(a) \in v \cap f(a) \in f(A)$

$\Rightarrow f(a) \in v \cap f(A)$

$\Rightarrow v \cap f(a) \neq \emptyset$

$\Rightarrow$The open set v in y containing intersect $f(A)$.

Therefore $y \in \overline{f(A)}$

Therefore $f\overline{(A)} \subset \overline{f(A)}$

(ii)$\Longrightarrow$(iii)

Assume that,$f\overline{(A)} \subset \overline{f(A)}$ $\forall$ A is a subset of X.

Let B be closed in y.

To prove:

$f^{-1}(B)$is closed in X.

Let A=$f^{-1}(B) \rightarrow (A)$

$f(A) = B \rightarrow (B)$

To Prove:

A is closed in X.

(ie) to prove A=$\overline{(A)}$

We know that,

A is a subset of $\overline{(A)} \rightarrow (1)$

Claim,

$\overline{(A)}$is a subset of A

[since$f\overline{(A)} \subset f\overline{(A)}$]

Let $x \in \overline{(A)} \Longrightarrow f(x) \in f\overline{(A)}$

$\Longrightarrow f(x) \in \overline{f(A)}$

$\Longrightarrow f(x) \in \bar{B}$

$\Longrightarrow f(x) \in B$

$\Longrightarrow x \in f^{-1}(B)$

$\Longrightarrow x \in A$

Therefore $\bar{A} \subset A \rightarrow (2)$

From (1) & (2)

$A = \bar{A}$

Therefore A is closed in X.

(ie)$f^{-1}(B)$ is closed in X.

(iii)⟹(i)

Assume that, B is closed in y.

$\Longrightarrow f^{-1}(B)$is closed in X.

To Prove:

f is continuous

v be open in y.

$\Rightarrow y - v$is closed in y.

$\Rightarrow f^{-1}(y - v)$is closed in x.

$\Rightarrow f^{-1}(y) - f^{-1}(v)$is closed in x.

$\Rightarrow f^{-1}(v)$isopen in x.

$\Rightarrow f$is continuous

(i)$\Rightarrow$(iv)

Assume that,

$f$is continuous

To prove:

For each $x \in X$ and each neighbourhood v of f(x) there isneighbourhood u of x such that $f(u) \subset v$.

Let $x \in X$

Let v be a neighbourhoodof$f(x)$.

To Prove:

$$f(u) \subset v$$

[therefore$f$ is continuous]

U=$f^{-1}(v)$ is a neighbourhood of X.

$f(u)$is a subset of v

(ie) $f(u) \subset v$

(iv)$\Rightarrow$(i)

Assume that,

For each $x \in X$ and each neighbourhood v of $f(x)$, there is neighbourhood v of x, such that $f(u) \subset v$.

To Prove:

f is continuous.

Let v be an open set of v.

Let x be a point of $f^{-1}(v)$.

$$\Longrightarrow x \in f^{-1}(v)$$

$$\Longrightarrow f(x) \in v$$

So that by hypothesis,

There is an neighbourhood u_x of x such that $f(u_x)) \subset v$.

Then $u_x \subset f^{-1}(v)$

$\Longrightarrow f^{-1}(v)$ is union of open set u_x so that

It is open

Hence f is continuous.

Homeomorphism:

Let X and Y be topological space .

Let $f: X \longrightarrow Y$ be a bijection. If both the function f and the inverse function.

$f^{-1}: Y \longrightarrow X$ are continuous then f is called a Homeomorphism.

Definition:

Given X such that

$$X = \Pi X_i$$

or the (possibly infinite) Cartesian product of the topological spaces X_i, indexed by i, the **box topology** on X is generated by the base

$$B = \{\Pi U_i / U_i \text{ is open}\}$$

The name *box* comes from the case of $\mathbf{R}^n$, in which the basis sets look like boxes. The set endowed with the box topology is sometimes denoted by

Box topology on $\mathbf{R}^\omega$:

- The box topology is completely regular

- The box topology is neither compact nor connected
- The box topology is not first countable (hence not metrizable)
- The box topology is not separable
- The box topology is paracompact (and hence normal and completely regular) if the continuum hypothesis is true

SUMMARY

- Topology is sometimes called "rubber-sheet geometry" because the objects can be stretched and contracted like rubber, but cannot be broken.
- In mathematics, the study of the properties of a geometric object that remains unchanged by deformations such as bending, stretching, or squeezing but not breaking.

QUESTIONS

1. Define Hausdroff Space
2. Define the K topology on the real line.
3. Show that the intersection of two topology is again a topology and the union need not be a topology
4. Let B and B’ be a basis for the topology $\tau\ and\ \tau'$ respectively on X then the following are equivalent (a) τ' is finer than τ (b) for each x in X and each basis element B ∈ Ḃ X there is a basis element B’ Ƀ such that x ∈ B’ Ƀ.
5. Define the subspace topology. For a subset Y of a topological space X, Define $\tau_y = \{Y \cap U / U \in \tau\}$. Prove that τ_y is a topology.
6. Show that every finite point set in a Hausdroff space X is closed.
7. Let X be a topological space. Prove that the following conditions holds

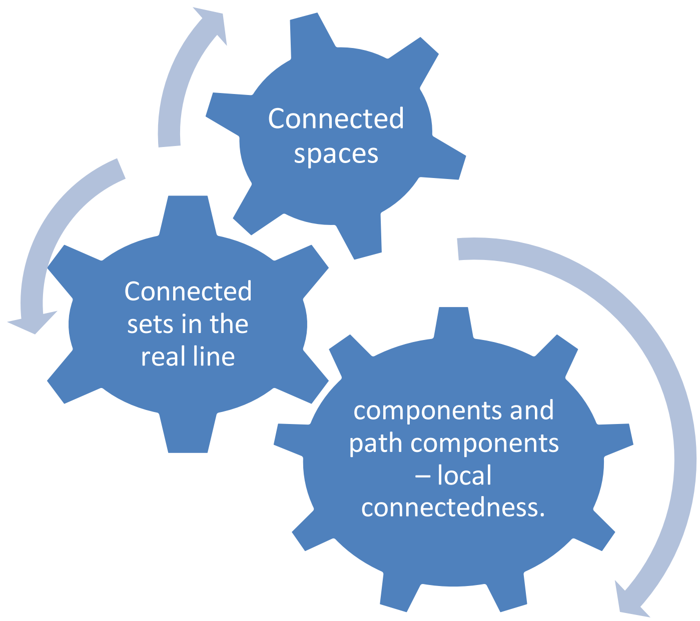
(i) ∅ and X are closed.

(ii) Arbitrary intersection of closed sets are closed.

(iii)Finite intersection of closed sets are closed.

8. Every simply ordered ser is a Hausdroff space in the order topology.(i) A subspace of Hausdroff space is Hausdroff space. (ii) The product of two Haudroff space is a Hausdroff space.

Unit III


Connected spaces
Connected sets in the real line
components and path components – local connectedness.

Content

- Connected spaces
- Connected sets in the real line
- Pasting Lemma
- Components and path components
- Local connectedness

<u>DEFINITION</u>:

A topological space (X, T) is said to be disconnected if there exist disjoint nonempty subsets $A, B \subseteq X$ such that $X = A \sqcup B$, and $\overline{A} \cap B = A \cap \overline{B} = \emptyset$.

If (X, T) is not disconnected, it is said to be connected. Just like with compactness we will often refer to subsets of topological spaces being connected, and in doing so we mean that the subset with its subspace topology is connected. Before going on, we state some of the many equivalent forms of this definition. The proof that these are all equivalent is basically immediate.

THOREM: The following are equivalent for a topological space (X, T).

1. (X, T) is disconnected.

2. There exist nonempty, disjoint, open sets $A, B \subseteq X$ such that $X = A \sqcup B$.

3. There exist nonempty, disjoint, closed sets $A, B \subseteq X$ such that $X = A \sqcup B$.

4. There is a nontrivial closed subset of X. That is, there is a subset $A \subseteq X$ that is both open and closed, and A is not X or $\emptyset$.

<u>THEOREM</u> :

<u>THE PASTING LEMMA</u>.

Let X and Y be topological space. Let $X = A \cup B$ where

A and B are closed in X. Let $f : A \to X$ and $g : B \to Y$ be

continuous. If $f(X) = g(X)$ for every $X \in A \cap B$ then f and g

combined to give a continuous function $h : X \to Y$ defined

by setting $h(X) = f(X)$ if $X \in A$ and $h(X) = g(X)$ if $X \in B$.

<u>PROOF</u>:

Given $X = A \cup B$ and A, B are closed in X.

$f : A \to X$ and $g : B \to Y$ are continuous.

Let $h : X \to Y$ by $h(x) = f(x)$ for all $x \in A$ and $h(x) =$

$g(x)$ for all $x \in B$.

$f(x) = g(x)$. for all $x \in A \cap B$.

To prove : $h : X \to Y$.

Let C be a closed set in Y.

To prove : $h^{-1}(c) = f^{-1}(c) \cup g^{-1}(c)$.

Let $x \in h^{-1}(c)$.

$\Leftrightarrow h(x) \in C$.

$\Leftrightarrow f(x) \in C$, for all $x \in A$.

$\Leftrightarrow x \in f^{-1}(c)$.

(Or)

$\Leftrightarrow x \in g^{-1}(C)$.

$\Leftrightarrow x \in f^{-1}(C) \cup g^{-1}(C)$.

$\therefore h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$.

$\therefore$ C is closed set in Y and $f : A \to x$ is continuous.

$\Rightarrow f^{-1}(C)$ is closed in A.

$f^{-1}(C)$ is closed in A and A is closed in X.

$\Rightarrow f^{-1}(C)$ is closed in X.

Similarly $g^{-1}(C)$ is closed in X.

$\therefore f^{-1}(C) \cup g^{-1}(C)$ is closed in X.

(ie) $h^{-1}(C)$ is closed in X.

$\therefore$ h is continuous.

<u>DEFINITION</u>:

Let J be an index set. Given a set X. We define a

J- tuple of elements of X to be a function $X : J \to X$. If $\propto$ is an

Element of J, we often denote the value of X at $\propto$ by $X \propto$,

rather than $X(\propto)$ is called if the $\propto$th coordinate of X.

And we often denote the function X itself by the

symbol $(X_\propto)_{\propto \in J}$ which is as close as we can come to a

" tuple notation " for an arbitrary index set J. We denote the

set of all J- Tuples of elements of X by X^J.

<u>THEOREM</u> :

<u>MAPS INTO PRODUCTS</u>.

Let X and Y be topological space. Let $f: A \to X \times Y$ be

Given by the equation $f(a) = (f_1 (a), f_2 (a))$ then f is continuous iff the functions $f_1 :A \to X$; $f_2 : A \to Y$ are continuous.

<u>PROOF</u>:

Given : $f : A \to X \times Y$.

$f (a) = (f_1 (a), f_2 (a))$.

Assume that f is continuous.

To prove : $f_1 : A \to X$, $f_2 : A \to Y$ are continuous.

Consider $\pi_1 : X \times Y \to X$ by $\pi_1(x, y) = x$.

To prove : π_1 is continuous.

Let U be open in X.

To prove : $\pi_1^{-1} (U)$ is open in $X \times Y$.

$\pi_1 (U \times Y) = U$.

$\Rightarrow \pi_1^{-1} (U) = U \times Y$.

U is open in X and Y is open in Y.

$\Rightarrow U \times Y$ is open in $X \times Y$.

$\pi_1^{-1} (U)$ is open in $X \times Y$.

π_1 is continuous.

$\pi_1 : X \times Y \to X$ are continuous.

$\therefore \pi 1$ of : A $\to$ X is continuous.

Now (π1 of) (a) = π1 (f (a)).

= π 1 (f1 (a) , f2 (a)).

(π1 of) (a) = f1 (a), for all a $\in$ A.

π1 of = f 1.

$\therefore \pi 1$ of : A $\to$ X is continuous.

$\Rightarrow$f1 : A $\to$ X is continuous.

Similarly $\Rightarrow$ f2 : A $\to$ Y is continuous.

Conversely,

Assume that f1 : A $\to$ X and f2 : A $\to$ Y are continuous.

To prove : f : A $\to$ X x Y is continuous.

Let U x V be open in X x Y.

$\Rightarrow$ U is open in X and V is open in X.

To prove : f^ -1 (U x V) is open in A.

First we prove that : f^-1 (U x V) = f1 ^ -1 (U) f2 ^ -1 (V).

Let a$\in$ f^-1 (U x V) $\Leftrightarrow$ f (a) $\in$ U x V.

$\Leftrightarrow$(f1 (a), f2 (a)) $\in$ U x V.

$\Leftrightarrow$ f1 (a) $\in$ U, f2 (a) $\in$ V.

$\Leftrightarrow$a$\in$ f1 ^-1 (U) , a $\in$ f2 ^ -1 (V).

f ^- 1 (U x V) = f1 ^ -1 (U) $\cap$ f2 ^ -1 (V).

U is open in X and f1 : A $\to$ X is continuous.

$\Rightarrow$ f1 ^ -1 (U) is open in A.

V is open in Y and f2 : A $\to$ Y is continuous.

$\Rightarrow f2 \wedge -1 (V)$ is open in A.

$\therefore f1 \wedge -1 (U) \cap f2 \wedge -1 (V)$ is open in A.

(ie) . $f 1 \wedge -1 (U \times V)$ is open in A.

$\therefore$ f is continuous.

DEFINITION:

Let $\{ A \propto \} \propto\in J$ be an indexed family of sets. Let $X = \cup$

$A \propto$.The Cartesian product of this indexed family , denoted by

$\pi A \propto$ is defined to be the set of all J – tuples $(X \propto) \propto\in J$ of

elements of X such that $X \propto\in A \propto$ for each $\propto\in J$.

(ie). It is the set of all functions $X : J \rightarrow \cup A \propto$ such that

$X (a) \in X \propto$ for each $\propto\in J$.

DEFINITION :

Let $\{ X \propto \} \propto\in J$ be an indexed family of topological

spaces. Let us take as a basis for a topology on the product

Space. $\pi X \propto$ the collection of all sets of the form $\pi U \propto$ where

$U \propto$ is open in $X \propto$ for each $\propto\in J$. The topology generated by

this basis is called the box topology.

SUMMARY

- A sphere is topologically equivalent to a cube because, without breaking them, each can be deformed into the other as if they were made of modeling clay.
- It is used in many branches of mathematics, such as differentiable equations, dynamical systems, knot theory, and Riemann surfaces in complex analysis.

QUESTIONS

1. Define Box Topology

2. Define Product Topology

3. State and Prove Pasting Lemma

4. Prove that Composition of two continuous topological spaces is continuous.

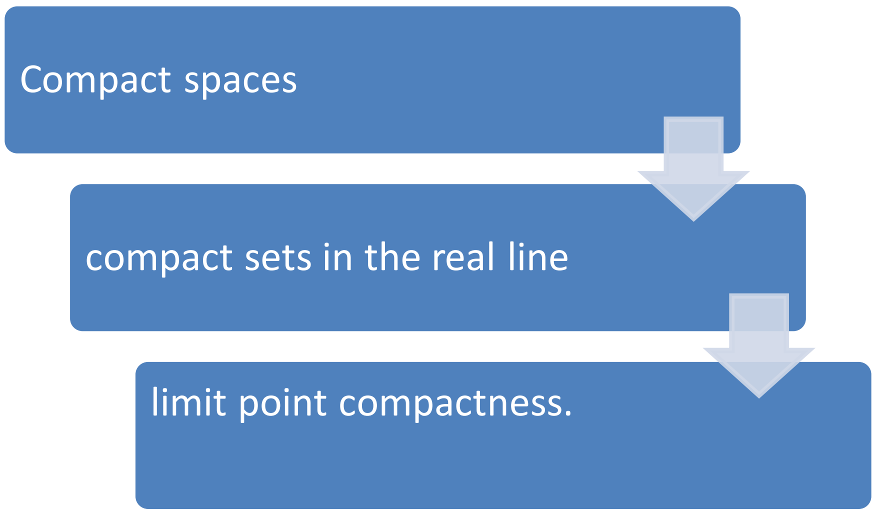
5. State and Prove Sequence Lemma

6. Let X be a metric space. Prove that $\bar{\bar{d}}$ is a metric that induced the same topology as d, where $\bar{\bar{d}}$ (x,y)= min {d(x,y), 1}

7. State and Prove Uniform limit theorem

8. Prove that the topological on R^n induced by the Euclidean metric d and the square metric are the same as the product topology on R^n

Unit IV


Compact spaces
compact sets in the real line
limit point compactness.

Content

- Compact spaces
- Compact sets in the real line
- Limit point compactness

Definition:

A topological space X is *compact* if every open cover has a finite sub cover. More precisely, if X=$\bigcup_{i \in I}$Ui for some collection {U_i : i∈I} of open sets indexed by a set I then there is a finite subset J⊂I such that X=$\bigcup_{i\in J}$Ui.

Theorem:

A) A subspace of a hausdroff space is hausdroff,a product of hausdroff space is hausdroff.
B) A subspace of a regular space is regular, a product of regular space is regular.

Proof:

a) Let x be a hausdroff space and let x and y be two points of the subspace y of x.

If U and V are disjoint neighbourhood in X of x and y respectively.

Then,U∩Y and V∩Y are disjoint neighbourhood of x and y in Y.

Hence a subspace of hausdroff space is hausdroff.

Let{x_α} be a family of hausdroff space

Let X={x_α} and Y={y_α} be distinct points of the product space$\pi_{x\alpha}$.

Since x$\neq y$ there is some index β, such that$x_\beta \neq y_\beta$,choose disjoint open sets U and V in containing$x_\beta$andy_β respectively.

Then the setsπ_β^{-1}(u) and $\pi_\beta^{-1}(v)$are disjoint open sets in$\pi_{x\alpha}$containing x and y respectively.

∴ product of hausdroff space is hausdroff.

b) Let y be a subspace of a regular space x
Wkt, Every regular space is hausdroff space
∴ x is hausdroff.
Wkt, subspace of a hausdroff space is hausdroff.
⇒ Y is hausdroff
Since y is hausdroff
Then one point sets are closed in Y.
Let x be a point of y and let B be a closed subset of y disjoint from x.

Now,$\overline{B} \cap Y = B$ where $\overline{B}$ denoted the closure of B in X.
∴ x$\overline{B}$ (since B is disjoint from x)
Since X is regular,we are can choose disjoint open sets U and V of x containing x and respectively.
∴ The subspace of the regular space is regular.
Let $\{x_\alpha\}$ be a family of regular space.
Since $\{x_\alpha\}$is regular we've $\{x_\alpha\}$ is hausdroff.
⇒π_x is hausdroff (since by (a))
Let x= $\{\pi_{x\alpha}\}$.
Since x is a hausdroff we've
The one point sets are closed in X.

To prove
⇒ X is regular
Let x=(x_α)be a points of X.
Let U be a neighbourhood of x in X.
Choose a basic elt $\pi_{U\alpha}$about x contained in U.
Choose for each α,a neighbourhood$V_\alpha$ofx_α inX_α such that

$$\overline{V_\alpha} \subset U_\alpha$$

If$U_\alpha = X_\alpha$ then V= $\pi_{V\alpha}$ is a neighbourhood of x in X
Since$\overline{V} = \pi_{\overline{V}\alpha}$
Then this gives$\overline{V} \subset \pi_{U\alpha} \subset U$
Then x is regular.
To prove this show that if$A_\alpha \subset X_\alpha$∀ αand if$\pi_{A\alpha} = A$ then$\bar{A} = \pi_{\bar{A}\alpha}$

For suppose that y=(y_α)in$\pi_{\bar{A}\alpha}$

Let U= U_α be is a basic elt containing Y
Since$y_\alpha \in \bar{A}_\alpha$
we've the open set$U_\alpha$must interest$A_\alpha$so we can choose a point$Z_\alpha \in A_\alpha \cap U_\alpha$,$\forall\ \alpha$
then U interests A in the point z= (z_α)
thus y∈ $\bar{A}$
ie)$\pi_{\bar{A}\alpha} \subset \bar{A} \to (1)$
conversely,
suppose that y is in$\bar{A}$ (y∈ $\bar{A}$)
we must such that,for any given index β we've$y_\beta \in \bar{A}_\beta$
thenπ_β-1(U_β) is a neighbourhood of y.
then its interests A is some points x.
thusU_β interests $\pi_\beta(A) = A_\beta$ in the points $\pi_\beta(z)$

$$\Longrightarrow y_\beta \in \bar{A}_\beta$$

Ie)$\bar{A}$ is a subset of$\pi_{\bar{A}\alpha}(\bar{A} \subset \pi_{\bar{A}\alpha}) \rightarrow (2)$
from eqation(1) and (2)

$$\bar{A} = \pi\bar{A}_{\alpha}$$

Remark:

SΩ and S are normal space its product SΩ and S is not

Normal Action:

Let x be a space. let G be a topological group an action of G on X is a continuous map$\alpha : G \times X \rightarrow X$such that denoting $\alpha(g \times x)\ by$ g.x one has

i) e.x = $\forall\ x \in X$
ii) $g_1(g_2$.x) = $(g_1.g_2)$x $\forall\ x \in X$ and $g_1,g_2 \in G$.

Orbit space:

Define x~g ·x ∀x and g, The resulting quotional space is defined by x⁄g and called the orbit space of the action x.

Theorem:

Every regular space with a countable map basic is normal.

Proof:

Let X be a regular space with a countable basic B.Let A and B be disjoint closed subset of x.

Since x is regular each point x of A has a neighbourhood U not intersecting B.

Using lemma-1

Choose a neighbourhood v of x such that$\bar{V} \in U$ choose an elt of B containing x and contained in v.

By choosing such a basic elt for each each x in A.

We contruct a countable covering of A by open sets whose closure doesn′t interest B.

Since thus covering of A is countable we can index it with the positive integers.

(say) let it be $\{U_n\}$

Similarly choose a countable collection $\{U_n\}$ of open sets covering B such that each $\bar{V}_n$ is disjoint from A.

The sets U=$U_{un}$and V=U_{Vn} are open sets containing A and B respectively.

But they need not be disjoint.

Now, we prove

U and V are disjoint as in the following manner

Given n, define $U_n{}'$ = U_n-

Clearly, each is open because it is a difference of an open set and a closed set

Similarly each open.

This collection {} cover A, because each xϵA belongs to for some n and x belongs to name of the sets .

Similarly , the collection cover B.

Now, the open sets and are disjoint,

For,ifx∈ $u' \cap v'$then x∈ $u_j{}' \cap v_k{}'$for j and k.

Suppose that j ≤ k

It follows from the define of $u_j$that x∈ $U_j$and

Since j ≤ k

It follows from the define ofv_j that x∉ $\overline{U}_j$.

⟹⟸ Arises if j∈ k

Theorem:

Every metrizable space is normal.

Proof:

Let x be a metrizable space with metric d.

Let A and B are disjoint closed subsets of x.

For each aϵA choose ϵa so that the ball B(a, ϵa) doesn′t interest B.

Similarly, for each bϵB choose ϵb so thatB(b, ϵb) doesn′t interest A.

Define U= B(a, ϵa/2) and v= B(b, ϵb/2)

Then U and V are open sets containing A and B respectively.

Now, we′ve to prove:

U and V are disjoint.

For $z \epsilon U \cap V$ then $z \epsilon B(a, \epsilon a/2) \cap B(b, \epsilon b/2)$ for some $a \epsilon A$ and $b \epsilon B$.

Then, by triangle inequality

$$d(a, b) \leq d(a, z) + d(z, b)$$

$$\leq \epsilon/2 + \epsilon/2$$

$$= \epsilon a + \epsilon b/2$$

If $\epsilon a \leq \epsilon b$ then $d(a, b) < \epsilon b$ then $B(b, \epsilon b)$ contains the points a,

If $\epsilon b \leq \epsilon a$ then $d(a, b) < \epsilon a$ then $B(a, \epsilon a)$ contains the points b.

Neither situation is possible

Since a and b are disjoint

Hence U and V are disjoint

∴ x is normal.

Theorem:

Every compact hausdroff space is normal.

Proof:

Let x be a compact hausdroff space,

Wkt, x is regular.

For if x is a point of x and B is closed set in x not containing x then B is compact.

(closed subset of a compact space is compact)

Ie) B is compact subsets of a hausdroff space x.

Then disjoint open sets containing x and B respectively.

To prove: x is normal.

By lemma (unit-4 , lemma-8)

For given disjoint closed sets A and B in x

Choose for each point a of A, disjoint open set U_a and V_a containing a and B respectively.

The collection $\{U_a\}$ covers A because A is compact.

Also, A may be covered by finitely many U_{a1},…….,U_{am}

Then U=u_{a1}∪……..Uu_{am} and V=v_{a1}∩………..∩v_{am} are disjoint open sets containing A and B respectively.

<u>TUBE LEMMA:</u>

<u>STATEMENT</u>:

Consider the product space x y where Y be a compact

Let $x_\circ \in y$

If N be an open set of XxY containing the slice in XxY, then there exists a tube in XxYcointaining this slice and contained in N.

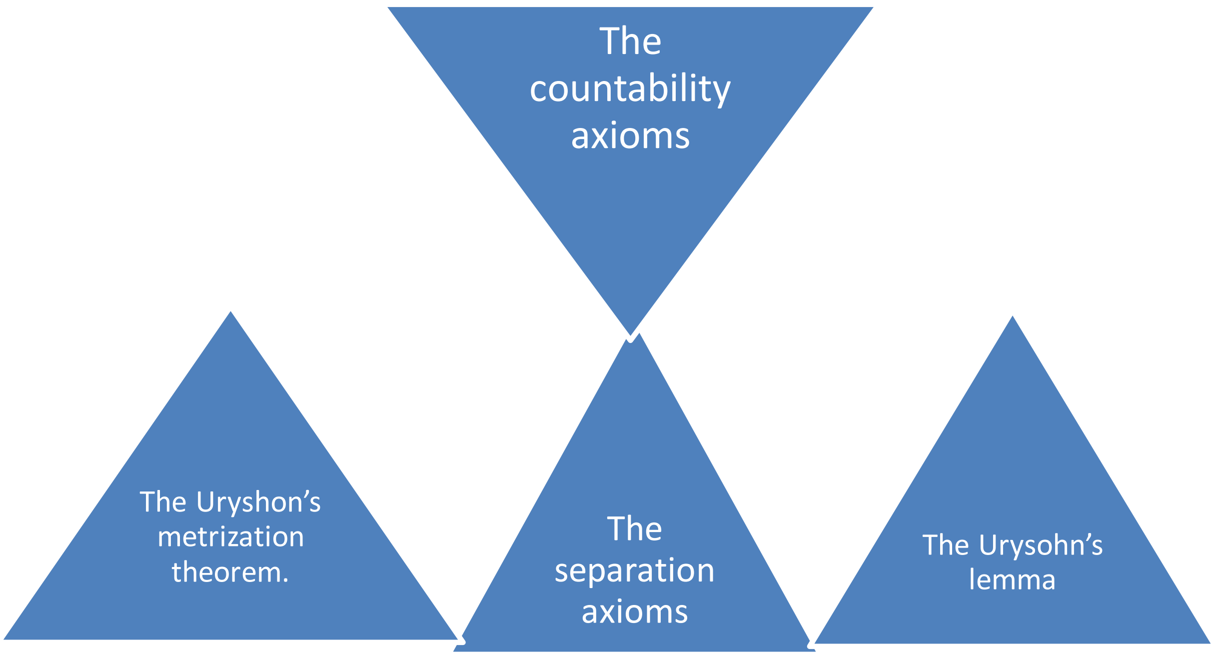
SUMMARY

- The tube lemma is useful to prove that the finite product of compact spaces is compact.

QUESTIONS

1. Define Compact Space. Give an example
2. Define limit point compact.
3. State and prove the extreme value theorem.
4. Show that every closed subspace of a compact space is compact.
5. State and prove Uniform continuity theorem.
6. Prove that every compact subspace of a Hausdroff space is closed.
7. State and prove Tube lemma.
8. Let X be a metrizable space prove that following are equivalent (a) X is compact (b) X is limit point compact (c) X is sequentially compact.

UNIT V


The countability axioms
The Uryshon's metrization theorem.
The separation axioms
The Urysohn's lemma

Content

- The countability axioms
- The separation axioms
- The Urysohn's lemma
- The Uryshon'smetrization theorem.

COUNTABILITY & SEPARATION AXIOMS

(§) The Countability Axioms:

Definition: Countablebasisat$x$:

A space$X$is said to have a **countable basis at$x$** if there is a countable collection$\mathbb{B}$of neighborhoods of x such that each neighborhood of x contains at least one of the elements of $\mathbb{B}$.

Definition: Firstcountable:

A space X that has a countable basis at each of its points is said to satisfy the **first countability axiom**, (or) to be **first-countable**.

Theorem: 1

Let X be a topological space.

a) Let A be a subset of X. If there is a sequence of points of A converging to x, then$x \in \bar{A}$; the converse holds if X is first-countable.

b) Let $f: X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converse holds ifX is firstcountable.

Proof:

The proof is a direct generalization of the proof.

Write Lemma:21.2(The sequencelemma) &Theorem:21.3 with proof.

(***Lemma 21.2 (The sequence lemma)***:

Let X be a topological space; let $A \subset X$. If there is a sequence of points of A converging to x, then $x \in \bar{A}$; the converse holds if X is metrizable.

Theorem 21.3:

Let $f: X \to Y$. If the function f is continuous, then for every convergent sequence $x_n \to x$ in X, the sequence $f(x_n)$ converges to $f(x)$. The converseholds if X is metrizable.**)**

Definition: Secondcountable:

If a space X has a countable basis for its topology, then X is said to satisfy the **second countability axiom**, (or) to be **second-countable**.

Example:

The real line $\mathbb{R}$ are has a countable basis.The collection of all open intervals *(a,b)* with rational end points.Likewise,$\mathbb{R}^n$ has a countable basis.The collection of all products of intervals having rational end points. Even$\mathbb{R}^\omega$ has a countable basis.The collection of all products$\prod_{n\in\mathbb{Z}_+} U_n$, where$U_n$ is an open interval with rational end points for finitely many values of n, and$U_n = \mathbb{R}$ for all other values of n.

Remark:

The second countability axiom$\Longrightarrow$The first countability axiom

Proof:

For if $\mathbb{B}$ is a countable basis for the topology of X, then the subset of$\mathbb{B}$ consisting of these basis element containing the point$x$ is a countable basis at X.

Theorem:

A subspace of a first countable space is first countable, and a countable product of first countable spaces is first countable. A subspace of a second countable space is second countable, and a countable product of second countable spaces is second countable.

Proof:

Write above remark.

Consider the second countability axiom.

If $\mathbb{B}$ is a countable basis for X,

then$\{B \cap A \mid B \in \mathbb{B}\}$ is a countable basis for the subspace$A$of X.

If$\mathbb{B}_i$is a countable basis for the spaceX_i,

then the collection of all products $\prod U_i$, where$U_i \in \mathbb{B}_i$ for finitely many values of $i$and

$U_i = X_i$forall other values of$i$, is a countable basis for$\prod X_i$.

The proof for the first countability axiom is similar.

Definition: **Dense**:

A subspace A of a space X is said to be **dense** in X if $\bar{A} = X$.

Theorem:

Suppose that Xhas a countable basis. Then:

a) Every open covering of Xcontains a countablesubcollectioncovering$X$.

b) There exists a countable subset of$X$that is dense in X.

Proof:

Given X has a countable basis.

Let $\{B_n\}$be a countable basis forX.

a. Let $\mathcal{A}$be an open covering of X.

By definition of countable basis,

For each positive integer $n \in Z_+$,we can choose an element of $A_n$of $\mathcal{A}$ containing the basis elementB_n.

The collection $\mathcal{A}'$ of the sets A_n is countable, because it is index with a subset J of the positive integers.

Also it covers X.

For given a point $x \in X$ we can choose an element A of $\mathcal{A}$ containing x.

Since A is open, there is a basis element B_n

such that $x \in B_n \subset A$.

Since B_n lies in an element of $\mathcal{A}$, the index n belongs to the set J.

So A_n is defined.

Since $A_n \subset B_n$, it contains x.

Thus $\mathcal{A}'$ is a countable subcollection of $\mathcal{A}$ that covers X.

b. From each non empty basis element $B_n$choose a point x_n.

Let Dbe the set$D = \{x_n | \, n \in \, Z_+\}$is dense in X.

Because given a point x of X,every basis element containing x intersects$D$,

So X belongs to $\overline{D}$.

Remark:

The space having a countable dense subset is often said to be **separable**.

Definition: Lindelöf space:

A space for which every open covering contains a countable subcovering is called a **Lindelöf space.**

The Separation Axioms:

Definition: Regular:

Suppose that one-point sets are closed in X. Then X is said to be **regular** if for each pair consisting of a point x and a closed set B disjoint from x, there exist disjoint open sets containing x and B, respectively.

Definition: Normal:

The space X is said to be **normal** if for each pair A, B of disjoint closed sets of X, there exist disjoint open sets containing A and B, respectively.

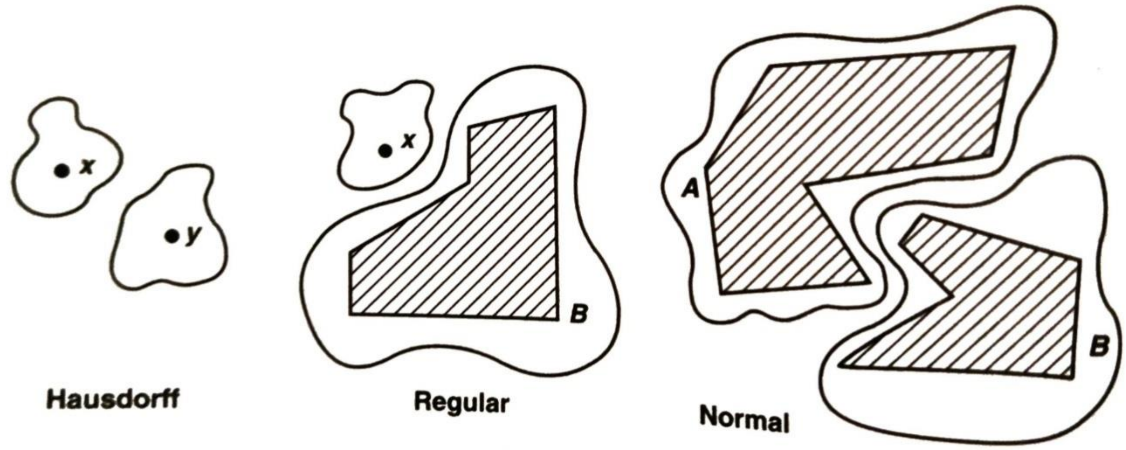


Figure 31.1

Result:

Every regular space is Hausdroff and every normal space is regular.

Normal $\Longrightarrow$ Regular $\Longrightarrow$ Hausdroff.

Lemma:

Let Xbe a topological space. Let one-point sets in Xbe closed.

a) $X$is regular if and only if given a point x of X and a neighborhood U of x, there is a neighborhood$V$of x such that $\bar{V} \subset U$.
b) X is normal if and only if given a closed set A and an open set Ucontaining$A$, there is an open set V containing$A$such that $\bar{V} \subset U$.

Proof:

a) Let Xbe regular.

Suppose that the point $x$and the neighborhood U of x are given.

Let $B = X - U$; then B is closed set, since X is regular there exist disjoint open sets U and W containing$x$and B, respectively.

The set $\bar{V}$ is disjoint from B,because if $y \in B$,then the set W is a neighborhood of y disjoint from V.

Therefore, $\bar{V} \subset U$.

Conversely,Suppose that the point x and the closed set B not containing x are given.

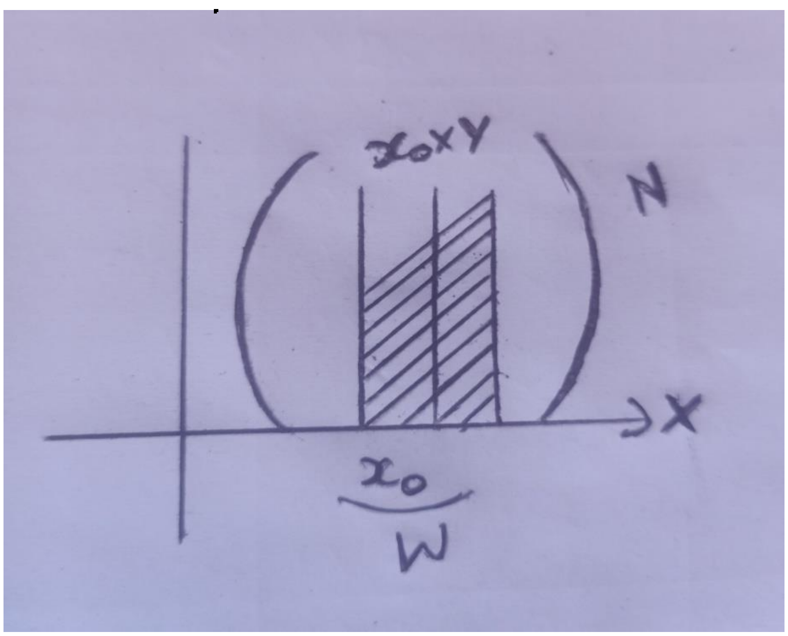
Let $U = X - B$.

By hypothesis, there is a neighborhood V of x such that $\bar{V} \subset U$.

Then the open sets V and $X - \bar{V}$ are disjoint open sets consisting x and B respectively.

Then X is regular.

b) Replace the point x by the set A through out proof of *(b)* follows.



PROOF:

Consider the vertical space slice X_0xY

{ It is a homomorphism to Y

Y is compact }$\Rightarrow X_0xY$ is compact

GIVEN,

N is open in X×Y

$X_0xY \subset N$

$\Rightarrow X_0Xy \in X_0xY \ \forall\ y \in Y$

$\Rightarrow X_0xY \in N \forall y \in Y$

{ N is open In XxY

$X_0 \times Y \in N\}$ $\Rightarrow$ then $\ni$ a basic element UxV

$\ni : X_0xY \in UxV \subset N$

Where U is open in X

V is open in Y

$\therefore$ The collection {UxY/U open in X

V is open in Y

$UxV \subset N$ } is an open cover for X_0xY

But X_0xY is compact

$\therefore$ Such that a finite subcollection$U_1xV_1, U_2xV_2 \ldots\ldots..UnxVn$

$\ni : X_0xY \subset (U_1xV_1) \cup \cdots\cdots\cdots \cup (UnxVn)$

Let $N = U_1 \cap U_2 \cap \cdots\cdots\cdots Un$

Each U_i (i=1,2$\cdots\cdots\cdots$n) is open in X

$\Rightarrow$ W is open in X

$X_0 \in U_i$ (i= 1,2$\cdots\cdots$n)

$\Rightarrow X_0 \in W$

TO PROVE:

$W x Y \subset N$

Let $(x,y) \in W x Y$

$\Rightarrow X \in W,\ y \in Y$

$X \in W \Rightarrow X \in U_1 \cap U_2 \cap \cdots\cdots \cap Un$

$\Rightarrow X \in U_i \quad \forall\ i=1,2 \cdots\cdots n$

$Y \in Y \Rightarrow X_0 \in Y \in X_0 x Y \subset (U_1 x V_1) \cup \cdots \cup (Un x Vn)$

$\Rightarrow X_0 \in Y \in (U_1 x V_1) \cup \cdots\cdot \cup (Un x Vn)$

$\Rightarrow X_0 x Y \in U_i x V_i$ for some i

$\Rightarrow y \in V_i$ for some i

$\therefore (x,y) \in U_i x V_i \qquad [\because U_i x V_i \subset W]$

$\Rightarrow (x,y) \in N$

$\therefore W x Y \subset N$

Theorem:

The product finitely many product space is compact

PROOF:

First we prove that the product by two compact space is compact.

Let X,Y be two compact space

TO PROVE:

XxY is compact

Let $\{A_a\} \alpha \in j$ be an open cover for XxY

Let $X_0 \in X$

consider the slice $X_0 x Y$

{$X_0 x Y$ is homomorphism to Y

And Y is compact} $\Rightarrow X_0 x Y$ is compact subset of XxY

Since {Aα} is an open cover for XxY

$\Rightarrow \{A\alpha\}$ is a cover for $X_0 xY$ by sets open in XxY

[By lemma]

$\therefore$ Э a finite subcollection $A\alpha_1, A\alpha_2 \cdots\cdots A\alpha n$

$\ni$: $X_0 xY \subset A\alpha_1 \cup A\alpha_2 \cup \ldots\ldots. \cup A\alpha n$

Let $N = A\alpha_1 \cup A\alpha_2 \cup \ldots. \cup A\alpha n$

Each $A\alpha_i$ is open in XxY$\Rightarrow$ N is open in XxY

$X_0 xY \subset A\alpha_1 \cup A\alpha_2 \cup \cdots\cdots\cdot \cup A\alpha n = N$

$\Rightarrow X_0 xY \subset N$

By tube lemma,

$\therefore$Such that a neighbourhood say X_0 in X $\ni$: $W \times Y \subset N$

$\Rightarrow WxY \subset A\alpha_1 \cup A\alpha_2 \cup \cdots\cdots \cup A\alpha n$

CONCLUSION:

Thus for each $x \in X$

We can find an open set W_x in X

$\ni$: (i) $X \in W_x$

(ii) $W_x xY$ can be cover by finite number of members of $\{A\alpha\}$

$\therefore$ The collection $\{W_x / x \in X\}$ is an open cover for X .But X is compact

$\therefore$ Such that a finite subcollection$Wx_1 \ldots\ldots Wxn$

$\ni$: $Wx_1 \cup Wx_2 \cup \ldots....UWxm = X$

$\Rightarrow (Wx_1 \cup Wx_2 \cup \ldots. \cup Wxm)\ XY = XxY$

$\Rightarrow (Wx_1\ XY) \cup \ldots\ldots. \cup (WxmXY) = XxY$

$\therefore$ Each $W_{xi} xY$ (i=1,2….W) can be covered by finite number of member of $\{A\alpha\}$

$\therefore (Wx_1 XY) \cup (Wx_2 XY) \cup \ldots.. \cup (Wx\ XY) = XxY$ can be covered by finite number of $\{A\alpha\}$

$\therefore$ XxY is compact

{If X is compact

Y is compact} ⇒XxY is compact

{If $X_1xX_2x\ldots\ldots xXn$ is compact

Xn is compact} ⇒ $(X_1xX_2x\ldots xX_{n-1})xXn$ is compact

But $X_1xX_2x\ldots\ldots xXn$ is homomorphic to $X_1xX_2x\ldots xXn$ is compact

∴ By Induction

If $X_1,X_2\ldots\ldots Xn$ are compact space then $X_1xX_2x\ldots\ldots xXn$ is compact.

FINITE INTERSECTION CONDITION :

A collection ζ of subset of X is said to satisfy the finite Intersection condition. If for every finite subcollection{ $C_1,C_2\ldots\ldots Cn$} of ζ

The Intersection $C_1\cap\cap C_2\cap\ldots\ldots\cap Cn=\varphi$

Let X be a topological space then X is compact ⇒ for every collection ς are closed set in X satisfy the finite Intersection condition ∩c∈ς C =φ

PROOF:

Let X be compact

Let ς be collection of closed set in X satisfying the finite Intersection condition.

TO PROVE:

$$\bigcap_{c\epsilon\zeta} \boldsymbol{C} \neq \boldsymbol{\Phi}$$

Suppose,

$$\bigcap_{C\epsilon\zeta} C \neq \Phi$$

$\Longrightarrow (\bigcap_{C\epsilon\zeta} C)^{c} \neq \Phi^{c}$

$\Longrightarrow \bigcup_{C\epsilon\zeta} C^{c} = X$

⟹ each

The collection { X-C/Cϵζ} is an open cover for X But X is compact.

∴∋an finite subcollection $X\text{-}C_1,X\text{-}C_2\ldots\ldots X\text{-}Cn$

$\ni : \bigcup_{i=1}^{n}(x - c_i) = X$

$\Longrightarrow (\bigcup_{i=1}^{n}(X - C_i))^c = X^c$

$\Longrightarrow \bigcap_{i=1}^{n} c_i = \emptyset$

Which is a contradiction (Since ζ satisfy the Finite Intersection collection)

$$\therefore \bigcap_{c\,\in\zeta} C \neq \emptyset$$

Conversly,

Assume that ζ be a collection of closed set in X satisfying the Finite Intersection collection

$\bigcap_{c\in\zeta} C \neq \varphi$

<u>TO PROVE:</u>

X is compact

Let A be an open cover of X

<u>TO PROVE:</u>

A has the finite subcover

Suppose,

A has no finite subcover

(i.e) $A_1 \cup A_2 \cup \ldots\ldots\ldots \cup An = X$

For any finite subcollection from A

$(A_1 \cup A_2 \cup \ldots\ldots\ldots \cup An)^c \neq X^c$

$\Longrightarrow A_1{}^c \cap A^{2\,c} \cap \ldots \ldots \cap An^c \neq \emptyset$

$\Longrightarrow \bigcap_{i=1}^{n} A_i{}^c \neq \emptyset$

$\Longrightarrow \bigcap_{i=1}^{n} (X - A_i) \neq \emptyset$

∴The collection of {X -A/A⊂A} of closed set satisfying the finite Intersection collection

By hypothesis,

$$\bigcap_{i=1}^{n}(X - A) \neq \emptyset$$

$(\bigcap_{A \in A}(X - A))^{c} \neq \emptyset$

$$\bigcup_{A \in A} A \neq X \quad \{\ A is\ an\ open\ cover\ for\ x\}$$

$$\Rightarrow\Leftarrow$$

Suppose ∋ a finite subcollection from ζ covering X

Hence X is compact.

SUMMARY

- Countable sets form the foundation of a branch of mathematics called discrete Mathematics
- The separation axioms are axioms only in the sense that, when defining the notion of topological paces one could add these conditions as extra axioms to get a more restricted notion of what a topological space is.
- Topology is also used in string theory in physics, and for describing the space-time structure of universe.

QUESTIONS

1. Define limit point compact.
2. Define Metrizable Space.
3. Define Quotient topology.
4. Define linear continum.
5. Define Path connected
6. Show that every closed subspace of a compact space is compact.
7. Let A be a connected subspace of X .If A⊆ B⊆$\bar{A}$, then B is also connected.
8. A space X is locally connected iff for every open set U of X, each component of U is open in X.
9. The components of X are connected disjoint subset of X, whose union is X such that each connected subsets of X intersecting only one of them.
10. State and Prove Intermediate value theorem.

11. The topologies on R^n induced by the Euclidean metric d and the square metric R are the same as the product topology on $R^{n.}$ That is $R^n$is Metrizable.
12. State and prove sequence lemma.
13. State and prove the extreme value theorem.

14. State and prove Urysohn lemma.
15. Let X be a metrizable space prove that following are equivalent (a) X is compact(b) X is limit point compact (c) X is sequentially compact.
16. The finite cartesian product of a connected space is connected.
17. If L is a linear continum in the order topology then L is connected and so are intervals and rays in L.